Int J Theor Phys (2008) 47: 2513–2522
DOI 10.1007/s10773-008-9685-2

Relativistic Confinement of Neutral Fermions with Partially Exactly Solvable and Exactly Solvable PT-Symmetric Potentials in the Presence of Position-Dependent Mass

Chun-Sheng Jia · Ping-Quan Wang · Jian-Yi Liu · Su He

Received: 8 November 2007 / Accepted: 31 January 2008 / Published online: 14 February 2008

Abstract The relativistic problems of neutral fermions subject to a new partially exactly solvable PT-symmetric potential and an exactly solvable PT-symmetric hyperbolic cosecant potential in $1+1$ dimensions are investigated. The Dirac equation with the double-well-like mass distribution in the background of the PT-symmetric vector potential coupling can be mapped into the Schrödinger-like equation with the partially exactly solvable double-well potential. The position-dependent effective mass Dirac equation with the PT-symmetric hyperbolic cosecant potential can be mapped into the Schrödinger-like equation with the exactly solvable modified Pöschl-Teller potential. The real relativistic energy levels and corresponding spinor wavefunctions for the bound states have been given in a closed form.

Keywords Dirac equation · PT-symmetric potential · Energy spectrum

1 Introduction

The Hermiticity of a potential $V(x)$ as a necessary condition for the reality of an energy spectrum has been relaxed by the non-Hermitian PT symmetry of the potential $V(x)$ [1]. Following the pioneering work of Bender and Boettcher [1], non-Hermitian PT-symmetric potentials and pseudo-Hermitian potentials [2–4] have been studied widely because of their intrinsic interest [5–35] and also for many applications in different research fields, such as

C.-S. Jia (✉) · J.-Y. Liu
State Key Laboratory of Oil and Gas Reservoir Geology and Exploitation, Southwest Petroleum University, Chengdu 610500, People's Republic of China
e-mail: chshjia@263.net

P.-Q. Wang
Petroleum Engineering Institute, Southwest Petroleum University, Chengdu 610500, People's Republic of China

S. He
Scientific Research Office, Southwest Petroleum University, Chengdu 610500, People's Republic of China

nuclear physics [36, 37], quantum field theories [38–40] and electromagnetic wave traveling in a planar slab waveguide [41]. A potential $V(x)$ is said to possess PT symmetry if the relation $V(-x) = V^*(x)$ or $V(\xi - x) = V^*(x)$ exists under the transformation of $x \to -x$ (or $x \to \xi - x$) and $i \to -i$, where P denotes parity operator (space reflection) and T denotes time reversal.

In recent years, some authors have extended the research fields for the PT-symmetric quantum systems with a constant mass to the non-relativistic PT-symmetric quantum systems with position-dependent mass [42–48] and relativistic PT-symmetric position-dependent effective mass quantum systems [49–53]. Systems with position-dependent mass have been found to be very useful in studying the physical properties of various microstructures, such as quantum dots [54], semiconductor heterostructure [55–57], quantum liquids [58], etc. The ordering ambiguity of the mass and momentum operators exists in the non-relativistic case [59]. However, it is usually expected that this ordering ambiguity should disappears in the relativistic ambiance. In this regard, some authors investigated the exact solutions of the Dirac equation and Klein-Gordon equation with position-dependent mass for some PT-symmetric potentials. In Refs. [49, 50], the authors proposed a scheme to construct a PT-symmetric potential with a real relativistic energy spectrum in the setting of the position-dependent effective mass Dirac equation in $1+1$ dimensions. In the case of the position-dependent mass distribution with linear and inversely linear form, the bound state solutions of the effective mass Dirac equation for a singular PT-symmetric potential have been studied [49]. By using the method proposed in Ref. [49], the relativistic problem of neutral fermions subject to PT-symmetric trigonometric potential in $1+1$ dimensions has been investigated [51]. Following up of the work [49], a new method has been proposed to construct the exactly solvable PT-symmetric potentials within the framework of the $(1+1)$-dimensional position-dependent effective mass Dirac equation with the vector potential coupling scheme [52]. In Ref. [53], Mustafa and Mazharimousavi investigated the spectrum of the $(1+1)$-Dirac equation with position-dependent mass and complexified PT-symmetric Lorentz scalar interactions.

In the present work, we construct two new PT-symmetric complex potentials, which are in the absence of bound states in the non-relativistic Schrödinger equation with a constant mass. However, we will show that these PT-symmetric potentials can trap neutral fermions in the setting of the Dirac equation with the vector potential coupling in the presence of a position-dependent double-well-like mass distribution and a symmetric mass distribution. In the case of the double-well-like mass distribution, the corresponding position-dependent effective mass Dirac equation can be mapped into the Schrödinger-like equation with the partially exactly solvable double-well potential [60]. For the present PT-symmetric hyperbolic cosecant potential, the corresponding position-dependent effective mass Dirac equation can be mapped into the Schrödinger-like equation with the exactly solvable modified Pöschl-Teller potential. In a series of papers [61, 62], de Castro and collaborators investigated the real kink-like potential and singular trigonometric tangent potential in the context of the $(1+1)$-dimensional Dirac equation with a constant mass in the background of a pseudoscalar potential coupling. These potentials are also absent of bound states in the non-relativistic theory because they give rise to the ubiquitous repulsive potentials. However, these potentials are able to confine neutral fermions in the setting of the Dirac equation with a pseudoscalar potential coupling in $1+1$ dimensions.

2 Position-Dependent Mass Effective Dirac Equation with a Vector Potential Coupling in a 1 + 1 Dimension

In the $(1+1)$-dimensional Dirac equation, one can greatly simplify the solutions by imposing a two-component approach, both the positive and negative energy solution states are retained without the added complication of spin [63]. The $(1+1)$-dimensional time-independent Dirac equation with position-dependent effective mass for a fermion coupled to a vector potential $V(x)$ reads

$$(\alpha p+\beta M(x)+V)\Psi(x)=E\Psi(x), \tag{1}$$

where E is the energy of the fermion, p is the momentum operator, $M(x)$ denotes the position-dependent effective mass, and α, β are 2×2 matrices satisfying $\alpha^2=\beta^2=1$, $\{\alpha,\beta\}=0$. The atomic units, $h/2\pi=\hbar=c=1$, are chosen. c is the velocity of light and h is the Planck constant. We use $\alpha=\sigma_3$ and $\beta=\sigma_1$, where σ_1 and σ_3 are Pauli matrices. Multiplying both sides of (1) by σ_1, one can explicitly write the Dirac equation (1) in the form of

$$\left[-i\frac{d}{dx}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}+V(x)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}+M(x)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right]\Psi(x)=E\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\Psi(x). \tag{2}$$

The spinor wavefunction $\Psi(x)$ has two components. We denote the upper and lower components by $\phi(x)$ and $\theta(x)$, respectively. Equation (2) can be decomposed into the following two coupled differential equations

$$-i\frac{d\theta}{dx}+[E-V(x)]\theta-M(x)\phi=0, \tag{3}$$

$$i\frac{d\phi}{dx}+[E-V(x)]\phi-M(x)\theta=0. \tag{4}$$

In Ref. [49], the vector potential is imposed in the form of a non-Hermitian imaginary potential

$$V(x)=\frac{i}{2}\frac{1}{M(x)}\frac{dM(x)}{dx}. \tag{5}$$

Performing the following local scaling for the upper spinor component

$$\phi(x)=\sqrt{M(x)}\varphi(x), \tag{6}$$

and substituting the expressions (5) and (6) into (3) and (4), we obtain the following Schrödinger-like equation satisfied by the new wavefunction $\varphi(x)$,

$$-\frac{d^2\varphi}{dx^2}+V_{eff}(x)\varphi=E^2\varphi, \tag{7}$$

where the effective potential $V_{eff}(x)$ is given by

$$V_{eff}(x)=M(x)^2. \tag{8}$$

Choosing an appropriate position-dependent mass distribution function $M(x)$ and using (5), we can construct a PT-symmetric potential which is exactly solvable or partially exactly solvable in the setting of the position-dependent effective mass Dirac equation with the vector potential coupling. By solving the Schrödinger-like equation (7), one can obtain the relativistic energy spectra and corresponding spinor wavefunctions.

In the presence of a time-independent scalar potential $S(x)$ and a vector potential $V(x)$, the $(1+1)$-dimensional time-independent Dirac equation with a zero mass (in $\hbar = c = 1$ units) reads

$$(\alpha p + \beta S(x) + V)\Psi(x) = E\Psi(x). \tag{9}$$

Imposing the vector potential $V(x)$ as that given in (5) and substituting it into (9), we can obtain a Schrödinger-like equation by using the same procedure which has been used in deducing equation (7),

$$-\frac{d^2\tilde{\varphi}(x)}{dx^2} + \tilde{V}_{eff}(x)\tilde{\varphi}(x) = E^2\tilde{\varphi}(x), \tag{10}$$

where the effective potential $\tilde{V}_{eff}(x)$ and new function are given by $\tilde{V}_{eff}(x) = S(x)^2$ and $\phi(x) = \sqrt{S(x)}\tilde{\varphi}(x)$, respectively. It is obvious that (10) is equivalent with (7). This equivalence shows that the scalar potential $S(x)$ plays the same role in the zero mass Dirac equation as the position-dependent mass $M(x)$ in the position-dependent effective mass Dirac equation with a vector potential coupling.

3 The PT-Symmetric Singular Potential

3.1 PT-Symmetric Partially Exactly Solvable Potential

In the first application, we choose the mass distribution in the following form

$$M(x) = M_0|A\cosh\alpha x - 1|, \tag{11}$$

where the dimensionless constant A is real and $0 < A < 1$. This mass distribution is of the double-well-like form. The top of the hump in the mass distribution is $M = M_0|A-1|$. With the aid of computer software such as MAPLE, we plot the explicit form of the mass distribution for few values of A in Fig. 1. Substituting (11) into (5) leads us to produce a non-Hermitian complex potential

$$V(x) = \frac{i}{2}\frac{\alpha A\sinh\alpha x}{A\cosh\alpha x - 1}. \tag{12}$$

This potential is singular and infinity at $x = \frac{1}{\alpha}\ln(\frac{1}{A} - \sqrt{\frac{1}{A^2} - 1})$ and $x = \frac{1}{\alpha}\ln(\frac{1}{A} + \sqrt{\frac{1}{A^2} - 1})$, and invariant under the change $\alpha \to -\alpha$ so that the results can depend only on $|\alpha|$. The potential (12) shows $V(-x) = V^*(x)$, thus, this complex potential possesses PT symmetry. In Fig. 2, the explicit form of the imaginary part for the PT-symmetric potential (12) has been plotted for $A = 1/2$. The PT-symmetric singular potential (12) is unable to trap a fermion in the PT-symmetric non-relativistic theory. Substituting (11) into (8) and using (7), we obtain a Schrödinger-like equation

$$-\frac{d^2\varphi}{dx^2} + M_0^2(A\cosh\alpha x - 1)^2\varphi = \tilde{E}\varphi, \tag{13}$$

Fig. 1 A plot of the double-well-like mass distribution (11) in the three cases: $A = 1/2$ (*red solid line*), $A = 1/3$ (*blue dotted line*) and $A = 1/5$ (*green dashed line*)

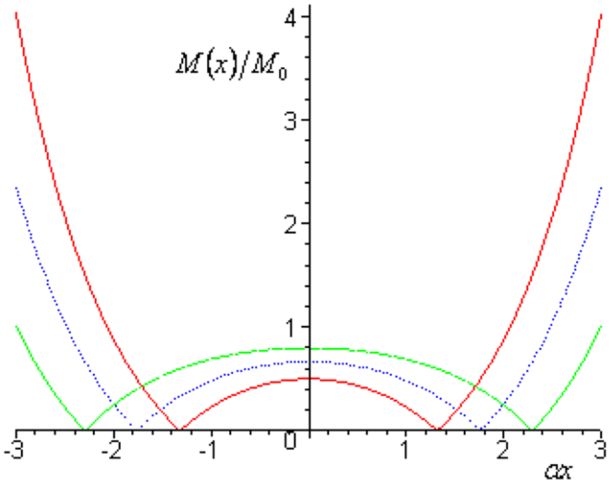
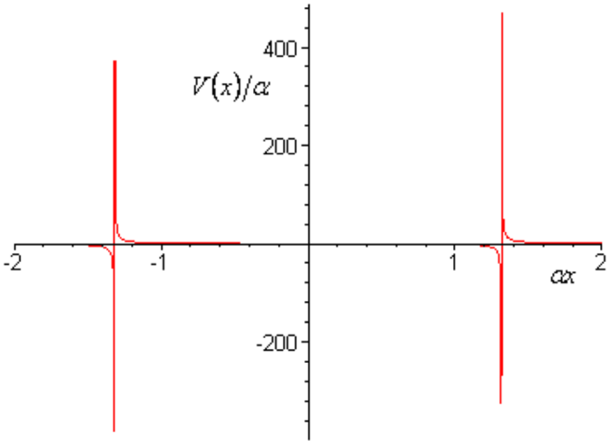


Fig. 2 A plot of the imaginary part of the PT-symmetric potential (12) for $A = 1/2$



where the effective energy is defined as $\tilde{E} = E^2$. Equation (13) shows that the position-dependent effective mass Dirac equation with the PT-symmetric potential (12) can be mapped into the partially exactly solvable Schrödinger-like equation with a partially exactly solvable double-well potential in $1+1$ dimensions. By using the su(2) Lie algebra approach, Konwent et al. [60] have solved the Schrödinger equation with the partially exactly solvable one-dimensional double-well potential. Choosing $M_0 = \frac{3}{2}\alpha$ and using the results given in Ref. [60], we obtain the corresponding three lowest effective energy eigenvalues

$$\tilde{E}_0 = \frac{9\alpha^2}{4}\left(A^2 + \frac{7}{9} - \frac{2}{9}\sqrt{1+36A^2}\right), \tag{14}$$

$$\tilde{E}_1 = \frac{9\alpha^2}{4}\left(A^2 + \frac{5}{9}\right), \tag{15}$$

$$\tilde{E}_2 = \frac{9\alpha^2}{4}\left(A^2 + \frac{7}{9} + \frac{2}{9}\sqrt{1+36A^2}\right). \tag{16}$$

In view of the expression $\tilde{E}=E^2$, we obtain the following three lowest relativistic energy levels for the PT-symmetric potential (12) in the setting of the Dirac theory with the vector potential coupling in the presence of the mass distribution given in (11),

$$E_0=\pm\frac{3|\alpha|}{2}\left(A^2+\frac{7}{9}-\frac{2}{9}\sqrt{1+36A^2}\right)^{1/2}, \tag{17}$$

$$E_1=\pm\frac{3|\alpha|}{2}\left(A^2+\frac{5}{9}\right)^{1/2}, \tag{18}$$

$$E_2=\pm\frac{3|\alpha|}{2}\left(A^2+\frac{7}{9}+\frac{2}{9}\sqrt{1+36A^2}\right)^{1/2}. \tag{19}$$

The discrete positive energy spectra and negative energy spectra are symmetric about $E=0$. This implies that the ways of the potential coupling to positive-energy component of the spinor and to the negative energy component of the spinor are the same. The effective eigenfunctions corresponding to the three lowest effective energy eigenvalues are given by [60]

$$\varphi_0(x)=\left(3A\cosh\alpha x-\frac{1-\sqrt{1+36A^2}}{2}\right)\exp\left(-\frac{3}{2}A\cosh\alpha x\right), \tag{20}$$

$$\varphi_1(x)=(\sinh\alpha x)\exp\left(-\frac{3}{2}A\cosh\alpha x\right), \tag{21}$$

$$\varphi_2(x)=\left(3A\cosh\alpha x-\frac{1+\sqrt{1+36A^2}}{2}\right)\exp\left(-\frac{3}{2}A\cosh\alpha x\right). \tag{22}$$

Substituting (20), (21) and (22) into (6) and making some algebraic manipulations, we obtain the unnormalized upper spinor wavefunctions corresponding to the three lowest energy levels,

$$\begin{aligned}\phi_0(x)&=\sqrt{\frac{3}{2}\alpha|A\cosh\alpha x-1|}\left(3A\cosh\alpha x-\frac{1-\sqrt{1+36A^2}}{2}\right)\\&\quad\times\exp\left(-\frac{3}{2}A\cosh\alpha x\right),\end{aligned} \tag{23}$$

$$\begin{aligned}\phi_1(x)&=\sqrt{\frac{3}{2}\alpha|A\cosh\alpha x-1|}(\sinh\alpha x)\\&\quad\times\exp\left(-\frac{3}{2}A\cosh\alpha x\right),\end{aligned} \tag{24}$$

$$\begin{aligned}\phi_2(x)&=\sqrt{\frac{3}{2}\alpha|A\cosh\alpha x-1|}\left(3A\cosh\alpha x-\frac{1+\sqrt{1+36A^2}}{2}\right)\\&\quad\times\exp\left(-\frac{3}{2}A\cosh\alpha x\right).\end{aligned} \tag{25}$$

With the help of the above three equations, we obtain the lower spinor wavefunctions corresponding to the three lowest energy levels from (4),

$$\theta_0(x)=\frac{1}{\frac{3}{2}\alpha|A\cosh\alpha x-1|}\left[E_0\phi_0(x)+i\left(1-\frac{3}{2}A\alpha\cosh\alpha x+\frac{1-\sqrt{1+36A^2}}{4}\right)\right.$$

$$\times 3A\alpha \sinh\alpha x\sqrt{\frac{3}{2}\alpha|A\cosh\alpha x-1|}\exp\left(-\frac{3}{2}A\cosh\alpha x\right)\Bigg], \tag{26}$$

$$\theta_1(x)=\frac{1}{\frac{3}{2}\alpha|A\cosh\alpha x-1|}\Bigg[E_1\phi_1(x)+i\alpha\left(\cosh\alpha x-\frac{3}{2}A\sinh^2\alpha x\right)$$
$$\times\sqrt{\frac{3}{2}\alpha|A\cosh\alpha x-1|}\exp\left(-\frac{3}{2}A\cosh\alpha x\right)\Bigg], \tag{27}$$

$$\theta_2(x)=\frac{1}{\frac{3}{2}\alpha|A\cosh\alpha x-1|}\Bigg[E_2\phi_2(x)+i\left(1-\frac{3}{2}A\alpha\cosh\alpha x+\frac{1+\sqrt{1+36A^2}}{4}\right)$$
$$\times 3A\alpha\sinh\alpha x\sqrt{\frac{3}{2}\alpha|A\cosh\alpha x-1|}\exp\left(-\frac{3}{2}A\cosh\alpha x\right)\Bigg]. \tag{28}$$

3.2 PT-Symmetric Hyperbolic Cosecant Potential

For the second example, we take the position-dependent mass distribution as a hyperbolic tangent function form in one spatial dimension,

$$M(x)=M_0|\tanh\alpha x|. \tag{29}$$

This mass distribution is of the symmetric form. The mass varies from the value $M=M_0$ for $x=-\infty$ to the value $M=0$ at $x=0$ and to the value $M=M_0$ for $x=+\infty$. We plot the explicit form of the mass distribution for $M_0=1$ in Fig. 3. Substituting (29) into (5), we obtain a PT-symmetric hyperbolic cosecant potential

$$V(x)=i\alpha\,\mathrm{cosech}\,2\alpha x. \tag{30}$$

This PT-symmetric potential is singular at $x=0$, which is unable to trap a fermion in the PT-symmetric non-relativistic theory. Substituting (29) into (8) and using (7), the Schrödinger-like (7) can be reduced to the form

$$-\frac{d^2\varphi}{dx^2}-M_0^2\,\mathrm{sech}^2\,\alpha x\varphi=\tilde{E}\varphi, \tag{31}$$

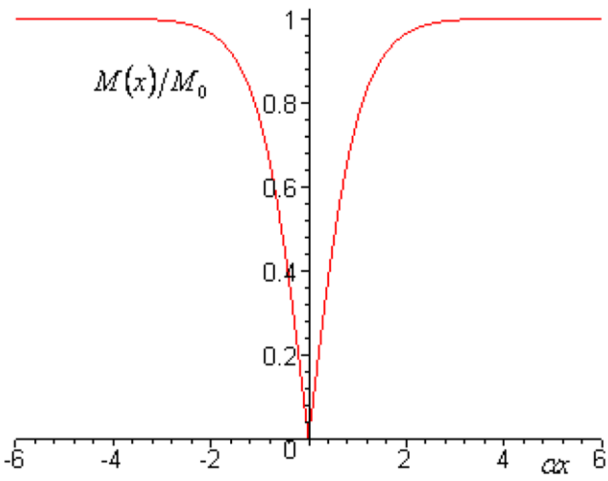


Fig. 3 A plot of the symmetric mass distribution (29)

where the effective energy $\tilde{E}$ is defined as $\tilde{E} = E^2 - M_0^2$. Equation (31) shows that the position-dependent effective mass Dirac equation with the PT-symmetric Lorentz vector potential (30) can be mapped into the Schrödinger-like equation with the exactly solvable modified Pöschl-Teller potential in $1+1$ dimensions. The solution of the relativistic problem can be obtained from a comparison with the non-relativistic solution of the modified Pöschl-Teller potential problem treated, for instance, in the reference [64]. The effective energy eigenvalues are given by [64]

$$\tilde{E}_n = -\alpha^2\left(n + \frac{1}{2} - \sqrt{\frac{1}{4} + \frac{M_0^2}{\alpha^2}}\right)^2, \tag{32}$$

where the quantum number $n = 0, 1, 2, \ldots$. With the help of the expression $\tilde{E} = E^2 - M_0^2$, we obtain the relativistic energy spectrum for the PT-symmetric hyperbolic cosecant potential (30) in the setting of the Dirac theory with the mass distribution given in (29),

$$E_n = \pm\left[M_0^2 - \alpha^2\left(n + \frac{1}{2} - \sqrt{\frac{1}{4} + \frac{M_0^2}{\alpha^2}}\right)^2\right]^{1/2}. \tag{33}$$

The discrete positive energy spectra and negative energy spectra are symmetric about $E = 0$. The effective eigenfunction corresponding to the effective energy eigenvalue $\tilde{E}_n$ is given by [64]

$$\varphi_n(x) = \left(\frac{1 + \tanh\alpha x}{2}\right)^{-p}\left(\frac{1 - \tanh\alpha x}{2}\right)^{-p} P_n^{-2p,-2p}(-\tanh\alpha x), \tag{34}$$

where $p = \frac{1}{2}(n + \frac{1}{2} - \sqrt{\frac{1}{4} + \frac{M_0^2}{\alpha^2}})$, $P_n^{-2p,-2p}(z)$ is the Jacobi polynomial. Applying the definition of the hyperbolic functions and making some algebraic manipulations, the effective eigenfunction $\varphi_n(x)$ reads,

$$\varphi_n(x) = (\cosh\alpha x)^{2p} P_n^{-2p,-2p}(-\tanh\alpha x). \tag{35}$$

From (6) and (35), we have the unnormalized upper spinor wavefunction corresponding to energy level E_n,

$$\phi_n(x) = \sqrt{M_0|\tanh\alpha x|}(\cosh\alpha x)^{2p} P_n^{-2p,-2p}(-\tanh\alpha x). \tag{36}$$

In order to make the upper spinor component $\phi_n(x)$ satisfying the asymptotic boundary condition, $\phi_n(\pm\infty) = 0$, the exponent of $\cosh\alpha x$ must be smaller than zero, i.e., $p < 0$. Further, we can guarantee the real energy spectra E_n in (33) if and only if the part in the square bracket of (33) is not negative. Consequently, we obtain the following restrictions for the quantum number n and the parameters α, and M_0,

$$n < \sqrt{\frac{1}{4} + \frac{M_0^2}{\alpha^2}} - \frac{1}{2}, \tag{37a}$$

$$M_0 \geq \alpha\left|n + \frac{1}{2} - \sqrt{\frac{1}{4} + \frac{M_0^2}{\alpha^2}}\right|. \tag{37b}$$

Applying the differential and recursion properties of the Jacobi polynomials, we can determine the lower spinor wavefunction corresponding to energy level E_n from (4),

$$\theta_n(x) = \frac{1}{M_0|\tanh\alpha x|}[(E_n + i\alpha(-n+2p)\tanh\alpha x)\phi_n(x) - i\alpha(n-2p)\phi_{n-1}(x)]. \quad (38)$$

4 Conclusion

In this work we have investigated the relativistic problems of neutral fermions subject to the PT-symmetric potentials (12) and (30) in $1+1$ dimensions. In the case of the mass distribution with the double-well-like form (11), the position-dependent effective mass Dirac equation with the vector potential coupling for the PT-symmetric potential (12) can be mapped into the Schrödinger-like equation with the partially exactly solvable double-well potential. We give the three lowest bound state energy levels and corresponding spinor wavefunctions in a closed form. In the second example, the position-dependent effective mass Dirac equation with the PT-symmetric hyperbolic cosecant potential (30) can be mapped into the Schrödinger-like equation with the exactly solvable modified Pöschl-Teller potential. By comparing the relativistic problem with the non-relativistic problem for the modified Pöschl-Teller potential, we obtain exactly bound state energy spectra and corresponding spinor wavefunctions. Both PT-symmetric potential (12) and PT-symmetric hyperbolic cosecant potential (30) are non-Hermitian and absent of bound states in the context of the non-relativistic Schrödinger equation with a constant mass, but they possess real discrete relativistic energy levels in the context of the position-dependent effective mass Dirac equation with the vector potential coupling.

Acknowledgements The authors would like to thank the referees for their helpful comments and suggestions which have improved the manuscript greatly. This work was supported by the National Natural Science Foundation of China under Grant No. 10675097, the Sichuan Province Foundation of China for Fundamental Research Projects under Grant No. 04JY029-062-2, the Natural Science Foundation of the Sichuan Education Bureau of China under Grant No. 07ZA140 and the Science and Technology Foundation of Southwest Petroleum University under Grant No. 2007XJZ073.

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